Further Dynamic-integrity-based Reliability Analysis

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Abstract. *Non-linear dynamical systems of different areas of engineering and technology are subject to parameters that should most of the times be realistically modelled as random variables, although they are usually considered to be deterministic. This paper addresses in a simple, almost naïve fashion, the reliability analysis of those systems, starting from a deterministic analysis, but then bringing into it the statistical properties of input variables. Although the concept of dynamical integrity has greatly contributed to establishing safe thresholds in dynamical systems, requiring that the basins of attraction should be robust, a reliability measure is still missing in that respect. In fact, supposing that the erosion curve of a dynamic integrity measure I (for instance, the integrity factor) has been obtained in terms of a system parameter A (for instance, a load amplitude, a load frequency, or still an imperfection parameter) using a deterministic approach, one can estimate the output statistical properties for* I*, in terms of those of the input parameter* A*, now considered as a random variable in its own right. Hence, once reference values for the integrity measure* I_{ref} *and the system parameter* A_{ref} *have been established, and assuming that increase of A beyond* A_{ref} *may prove to be dangerous, the probability that* $I \geq I_{ref}$*, provided that* $A \leq A_{ref}$*, would give a sound reliability assessment. A very simple approach towards this aim is proposed and applied herewith to an archetypal model of a rigid column asymmetrically constrained by a linear spring, subject to a conservative axial compression and a small dynamical transversal load, playing the role of a random dynamical imperfection. It should be said that this work is a continuation of another one, with the same archetypal model, yet taking into account only the effect of a statical imperfection. A versatile in-house code is used to obtain the basins of attraction and the erosion curves that give support to the proposed methodology. The reliability assessment is carried out in two different scenarios, namely varying either the dynamical imperfection amplitude or its frequency. The influence of nearness to either buckling, or to external resonance or even parametric resonance can be studied according to the proposed methodology. It is simple and easy to be applied. Therefore, one hopes that it can be absorbed in engineering design practice without its traditional resistance to incorporate new trends.* **Keywords:** Reliability analysis, Safe basin, Dynamic integrity, Integrity factor, Dynamic random imperfection

1. Introduction

In continuation to the ideas initially addressed in [1], we recast the same archetypal model studied in [2] of a rigid column asymmetrically constrained by a linear spring, subject to a conservative axial compression P and a small transversal load $Q(t) = Q_0 + Q_1 sin\hat{\Omega}t$, playing the role of a dynamical imperfection, as depicted in Fig.1. In [1], it was implicitly considered the case of a random statical imperfection $Q = Q_0$ only, which defined the so-called Koiter's load with certain statistical properties, whereas in the present paper either Q_1 or $\hat{\Omega}$ will be assumed as random variables.

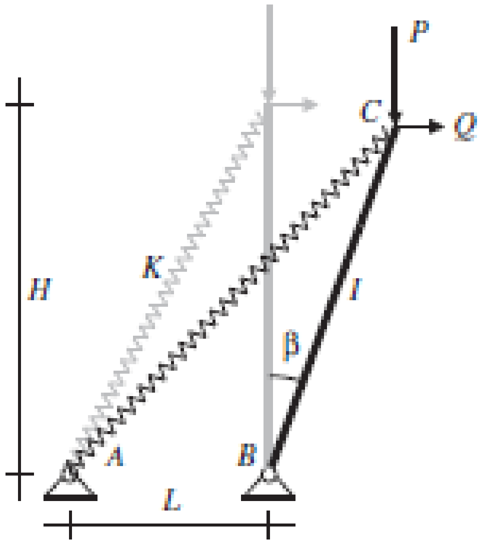


Figure 1. Archetypal model, adapted from [2].

In [2], the dimensionless equation of motion of this model is presented in the form:

$$\ddot{\beta} + c\dot{\beta} - psin\beta + \left[1 - \frac{1}{\sqrt{1+\alpha sin\beta}} - (q_0 + q_1 sin\widehat{\omega}\tau)\right] cos\beta = 0 \qquad (1)$$

where c stands for an assumed viscous damping coefficient, $p = \frac{P}{KL}$ is the dimensionless conservative compressive force, $\alpha = \frac{2LH}{(L^2+H^2)}$ defines the overall geometry, $q = \frac{Q}{KL} = q_0 + q_1 sin\widehat{\omega}\tau$ is the dimensionless lateral load playing the role of a statical plus a dynamical imperfection, $\widehat{\omega} = \widehat{\Omega}\sqrt{\frac{J}{KLH}}$ is the dimensionless forcing frequency, J is the rigid column mass moment of inertia with respect to hinge B, $\tau = t\sqrt{\frac{KLH}{J}}$ is the dimensionless time, and over dots mean differentiation with respect to τ.

It is worth mentioning that linearization of Eq.(1) indicates that the dimensionless natural frequency of the system is $\omega = \sqrt{\frac{\alpha}{2} - p}$, so that the system approaches buckling as p approaches $\frac{\alpha}{2}$ (Euler's load). In other scenarios, it approaches external or parametric resonances as $\widehat{\omega}$ approaches either ω or 2ω, respectively. Hence, approaching to either buckling or resonances could take place. By the way, parametric instability in this system is associated to quadratic nonlinearities. In this paper, only loads well below Euler's load will be considered, so that attention is focused on the resonances. In any case, basins of attraction will be obtained to identify attractors and evaluate the associated integrity measure I (it can be GIM, LIM or IF [3, 4]), for a set of system control parameters A (for instance, q_0, as in [1]; or q_1 and $\widehat{\omega}$ as herewith). Varying the chosen parameter A, the erosion curve $I(A)$ can be drawn, following a deterministic analysis. Yet, provided the statistical properties of the input parameter A are known, the statistical properties of the output variable I can be estimated [1, 5], thus allowing a reliability analysis to be made at different complexity levels [6]. What it will be presented here is perhaps the simplest, even naivest, reliability analysis of them all. It will be applied to the erosion profiles $IF(q_1)$ and $IF(\widehat{\omega})$.

2. Methodology for a simple reliability analysis

Consider a generic erosion profile $I(A)$. Its local slope in absolute value is given by $\left|\frac{dI}{dA}\right|$. Supposing that the parameter A is a Gaussian random variable, with a standard deviation σ_A about the expected value $\bar{A}$, it is assumed that the output I will also be a Gaussian random variable with a local standard deviation $\sigma_I = \sigma_A \left|\frac{dI}{dA}\right|$ about the expected value $\bar{I}$. This assumption has been crosschecked in [1] and shown to work out well. A simple way to consider safety in engineering design is to choose a minimum reference value I_{ref} to be achieved (with a previously defined probability to be attained), provided the parameter A would not be larger than a specified value A_{ref}. Hence, for every point $(\bar{A}, \bar{I})$, it can be assessed the probability for which the integrity measure complies with $I \geq I_{ref}$, provided $A \leq A_{ref}$.

2.1. A simple reliability analysis for the erosion profile $IF(q_1)$

The erosion profile of Fig.2 was extracted from [2], where the following fixed system parameters were considered: $c = 0.01$; $q_0 = 0.01$; $\alpha = 0.8$, $p = 0$ and $\hat{\omega} = 0.8$. It refers to the attractor $\beta = 0$ of the perfect system. Notice that the linearized natural frequency of the system is $\omega = \sqrt{\frac{\alpha}{2}} = 0.632$. Thus, the dynamical imperfection forcing frequency is non-resonant. Nevertheless, the increase of the dynamical imperfection amplitude q_1 causes an erosion of both the integrity factor IF and the global integrity measure GIM. Both IF and GIM were normalized with respect to their values for $q_1 = 0$. So, both erosion profiles depart from the same point and end up at the same point, which corresponds to the escape Koiter's load for q_1 [2]. However, in between the extreme points, it is noticed a clear difference between IF and GIM, which can be explained by the fractalization of the safe basin that is not fully detected by GIM.

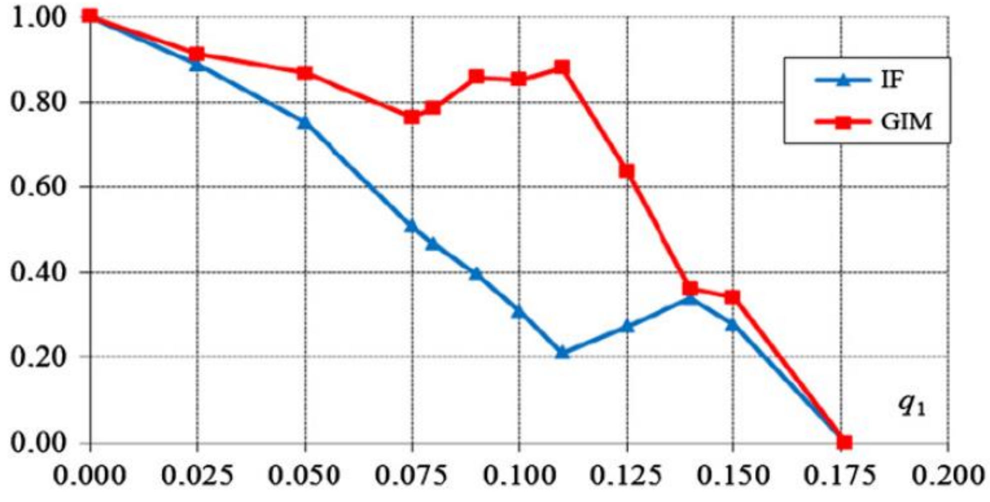


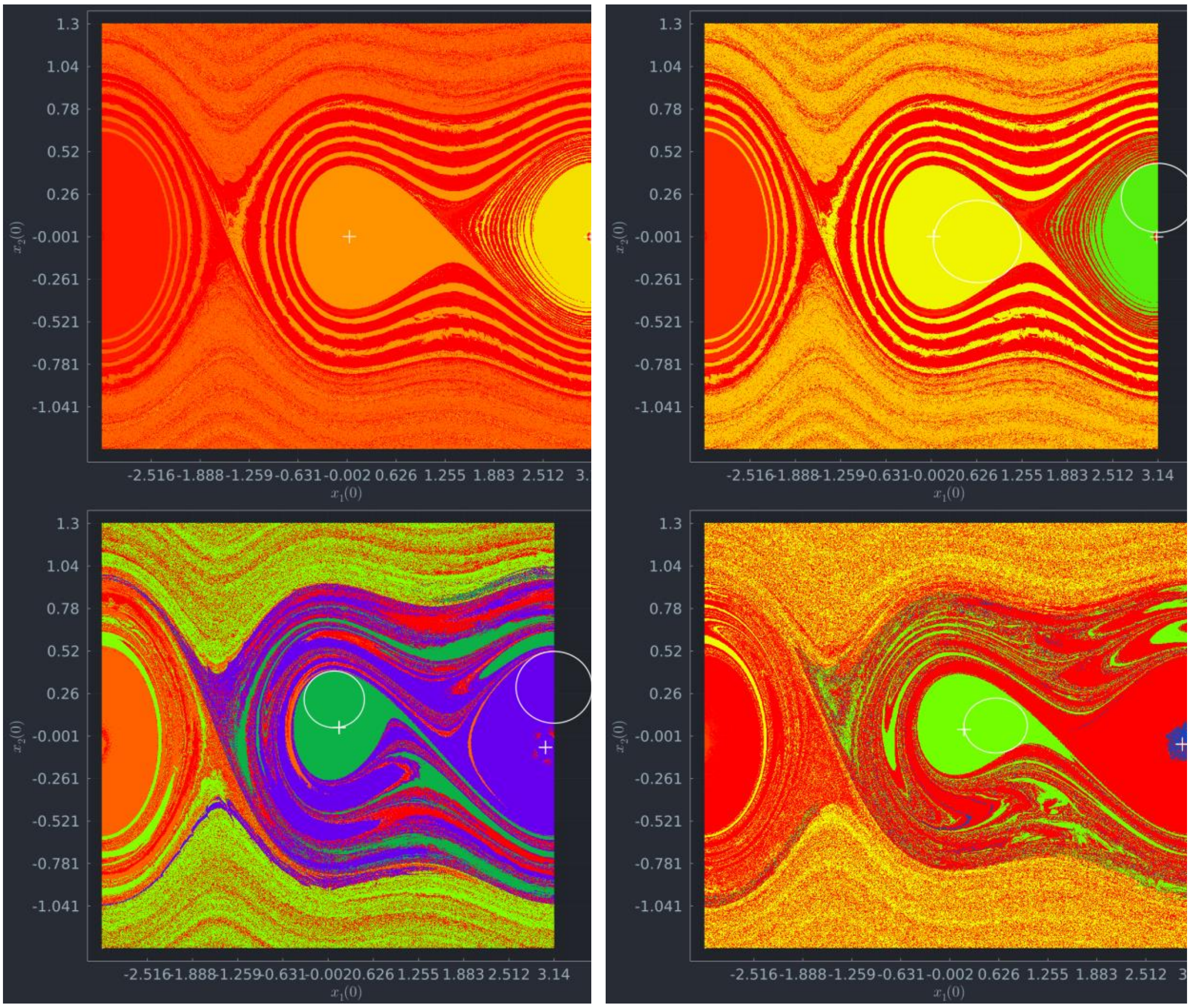
Figure 2. Erosion profiles $\boldsymbol{IF} \times \boldsymbol{q_1}$ and $\boldsymbol{GIM} \times \boldsymbol{q_1}$ extracted from Fig.11 of [2].

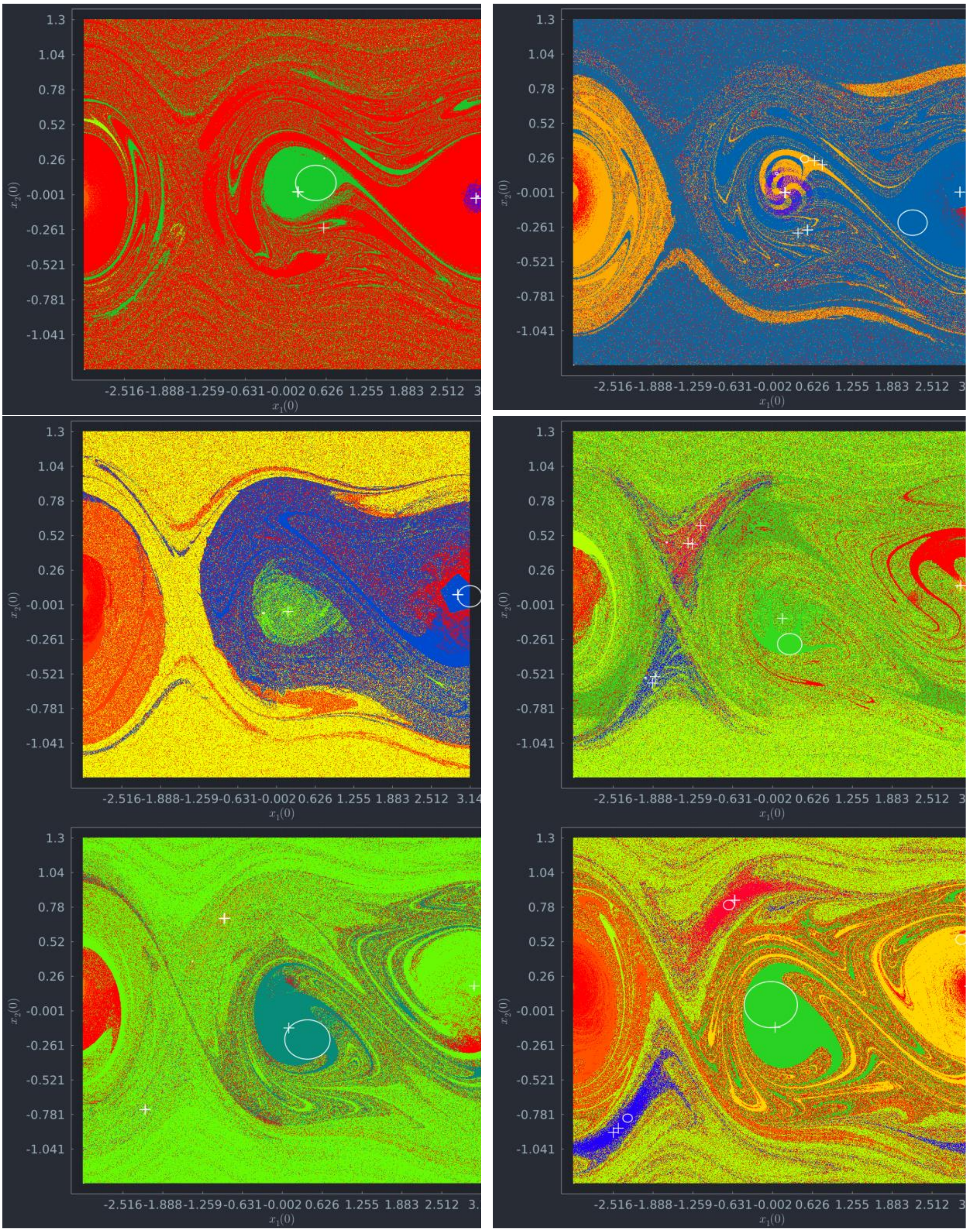
For the sake of an illustration, the simplified methodology to carry out the reliability analysis, as previously explained, is applied to the erosion profile $IF \times q_1$ of Fig.2, at the point $(\overline{q_1}, \overline{IF}) = (0.075; 0.50)$, for which $\left|\frac{dI}{dA}\right| \cong 9$. Assuming an input standard deviation $\sigma_{q_1} = 0.040$, the estimated output standard deviation would be $\sigma_{IF} = 0.36$, leading to 31.73% probability for IF to be at least $\overline{IF} + \sigma_{IF} = 0.86$, provided q_1 is lot larger than $\overline{q_1} + \sigma_{q_1} = 0.115$; 50% probability for IF to be at least $\overline{IF} = 0.50$, provided q_1 is lot larger than $\overline{q_1} = 0.075$; and 68.27% probability for IF to be at least $\overline{IF} - \sigma_{IF} =$

0.14, provided q_1 is lot larger than $\overline{q_1} - \sigma_{q_1} = 0.035$. These results could be used to decide whether the choices of $\overline{IF} = 0.50$, and henceforth $\overline{q_1} = 0.075$, were good enough for a safe engineering design.

2.2. A simple reliability analysis for the erosion profile $IF(\hat{\omega})$

The in-house *PoliBoA* code, based on [6], is fit to obtain the basins of attraction of dynamic systems within an n-dimensional phase space, as well as the desired dynamic integrity measure (GIM, LIM or IF). Fig.3 depicts the basins of attraction for a sample of values of the dynamical imperfection frequency $\hat{\omega}$, with the indication of the integrity factor IF associated with the radius of the maximum circle inscribed within the corresponding safe basin. The following system parameters were kept fixed: $c = 0.01$; $q_0 = 0.01$; $q_1 = 0.05$, $\alpha = 0.8$ and $p = 0.05$, whereas $\hat{\omega}$ assumes successively the values: 0.00, 0.10, 0.20, 0.30, 0.35, 0.40, 0.50, 0.60, 0.70, 0.80, 0.90, 1.00, 1.10, 1.20, 1.30 and 1.40, as one goes from left to right and from top to bottom in Fig. 3.





1.3
1.04
0.78
0.52
0.26
-0.001
-0.261
-0.521
-0.781
-1.041
$x_2(0)$
-2.516-1.888-1.259-0.631-0.002 0.626 1.255 1.883 2.512
$x_1(0)$

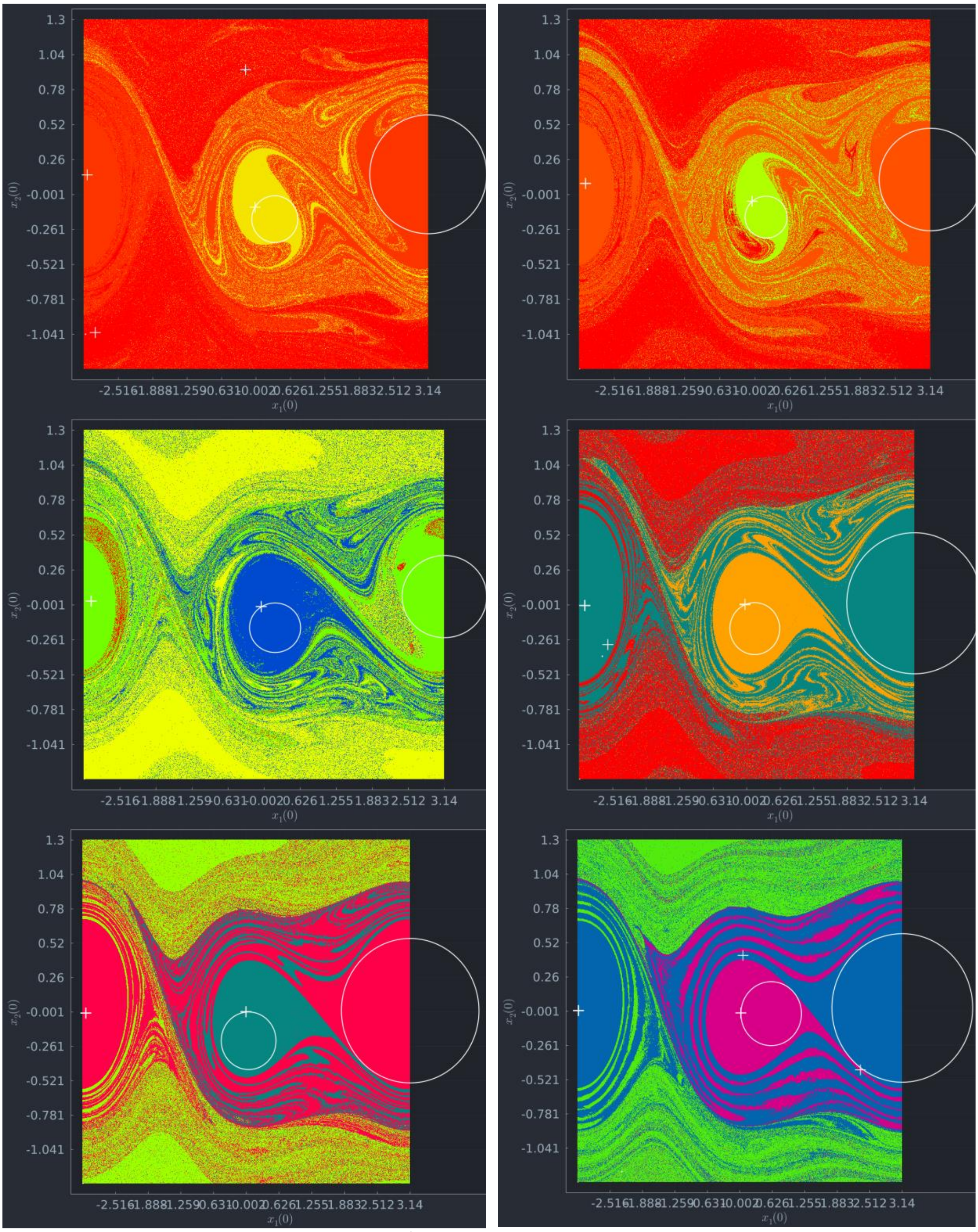


Figure 3. Basins of attraction in plane $\dot{\boldsymbol{\beta}} \times \boldsymbol{\beta}$ for several values of the imperfection frequency $\widehat{\boldsymbol{\omega}}$.

Hence, from the code Poli BoA it was possible to obtain the erosion profile $IF \times \widehat{\omega}$ of Fig. 4, considering the attractor $\beta = 0$ of the perfect system. The graph was normalized with respect to the IF for $\widehat{\omega} = 0$.

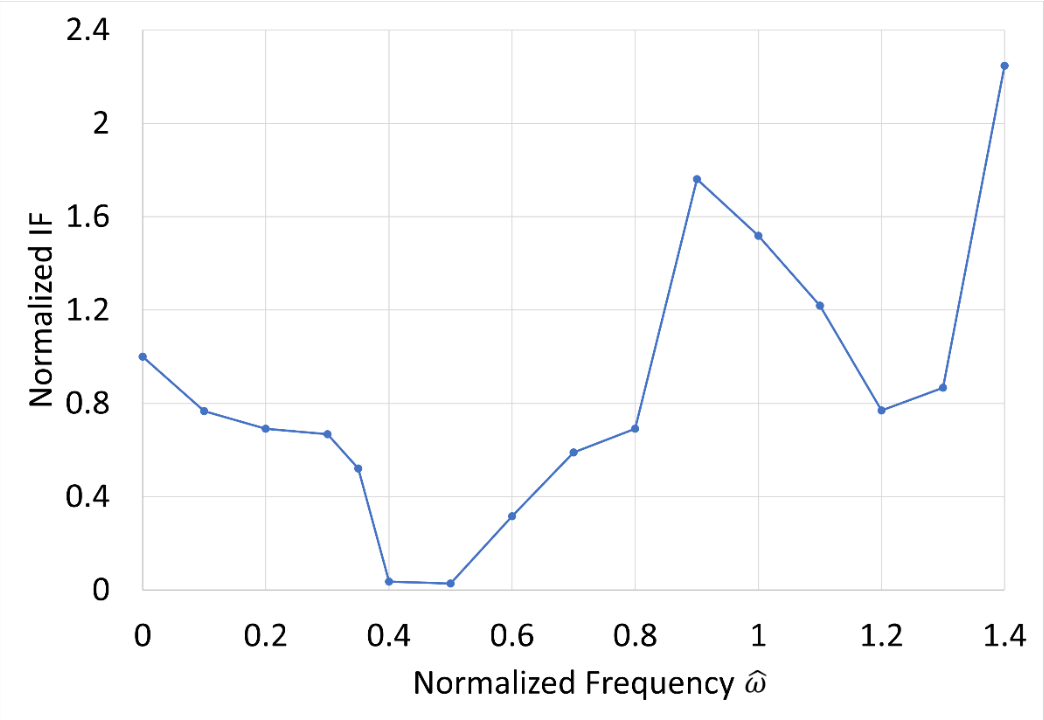


Figure 4. Erosion profile $\boldsymbol{IF} \times \widehat{\boldsymbol{\omega}}$ obtained with *PoliBoA* [3].

Notice that the system linearized natural frequency is $\omega = \sqrt{\frac{\alpha}{2} - p} = 0.5916$. IF is practically zero for $0.4 < \widehat{\omega} < 0.6$, range that includes the natural frequency, thus showing the effect of external resonance degrading the system dynamic integrity and, consequently, its reliability. For larger forcing frequencies, the system recovers its integrity factor, as the external resonance is left behind. However, a new steep loss in IF is detected, this time due to approaching the principal parametric resonance 2:1, for which $\widehat{\omega} = 1.1832$, again rising as parametric resonance is left behind.

A similar reliability analysis to that of Section 2.1 was carried out for $\widehat{\omega}$ modelled as a random variable. First, the analysis is illustrated for the point $\left(\overline{\widehat{\omega}}, \overline{IF}\right) = (0.35; 0.52)$, halfway towards the 1:1 external resonance, for which $\left|\frac{dI}{dA}\right| \cong 6.5$, as shown in Fig. 4. Assuming an input standard deviation $\sigma_{\widehat{\omega}} = 0.04$, the estimated output standard deviation would be $\sigma_{IF} = 0.26$, leading to 31.73% probability for $IF \geq \overline{IF} + \sigma_{IF} = 0.78$, provided $\widehat{\omega} \leq \overline{\widehat{\omega}} + \sigma_{\widehat{\omega}} = 0.39$; 50% probability for $IF \geq \overline{IF} = 0.52$, provided $\widehat{\omega} \leq \overline{\widehat{\omega}} = 0.35$; and 68.27% probability for $IF \geq \overline{IF} - \sigma_{IF} = 0.26$, provided $\widehat{\omega} \leq \overline{\widehat{\omega}} - \sigma_{\widehat{\omega}} = 0.31$. Again, these results could be used to decide whether the choices of $\overline{IF} = 0.52$ and $\overline{\widehat{\omega}} = 0.35$, were appropriate for a safe engineering design considering the external resonance. Next, looking at the parametric instability scenario, the analysis is illustrated for the point $\left(\overline{\widehat{\omega}}, \overline{IF}\right) = (1.10; 1.20)$, for which $\left|\frac{dI}{dA}\right| \cong 3.7$, as shown in Fig. 4. Assuming, as before, an input standard deviation $\sigma_{\widehat{\omega}} = 0.04$, the estimated output standard deviation would be $\sigma_{IF} = 0.15$, leading to 31.73% probability for $IF \geq \overline{IF} + \sigma_{IF} = 1.35$, provided $\widehat{\omega} \leq \overline{\widehat{\omega}} + \sigma_{\widehat{\omega}} = 1.14$; 50% probability for $IF \geq \overline{IF} = 1.20$, provided $\widehat{\omega} \leq \overline{\omega} = 1.10$; and 68.27% probability for $IF \geq \overline{IF} - \sigma_{IF} = 1.05$, provided $\widehat{\omega} \leq \overline{\widehat{\omega}} - \sigma_{\widehat{\omega}} = 1.06$. Again, these results could be used to decide whether the choices of $\overline{IF} = 1.20$ and $\overline{\widehat{\omega}} = 1.10$, were appropriate for a safe engineering design. The parametric resonance scenario does not seem to be critical in this simulation since its effect is upon the quadratic non-linearity.

3. Concluding remarks

The paper presents a simplified reliability analysis based on the erosion curves of a dynamical system, provided the statistical properties of an appropriate input random system parameter is known. The proposed methodology is applied, for the sake of an example, to the archetypal model of a rigid column hinged at the bottom and elastically restrained by an inclined linear spring, subject to a conservative compression and a lateral dynamic load that plays the role of an imperfection. Both the effects of randomness in the imperfection amplitude and frequency are discussed, emphasizing the effects upon the integrity measure, as dynamical thresholds, such as the Koiter's escape load or resonances (external or parametric) are approached. The simplicity and easiness required for the application of this methodology gives hope that this methodology can be absorbed in engineering design practice overcoming its traditional resistance to incorporate new trends. Reference should be made to an improved reliability analysis, as addressed in [1].

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